Published by Institute of Physics Publishing for SISSA/ISAS

Received: *June 30, 2003*
Accepted: *July 8, 2003*

The holonomy of the supercovariant connection and Killing spinors

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Abstract: We show that the holonomy of the supercovariant connection for M-theory backgrounds with N Killing spinors reduces to a subgroup of $\mathrm{SL}(32-N,\mathbb{R}) \ltimes (\oplus^N \mathbb{R}^{32-N})$. We use this to give the necessary and sufficient conditions for a background to admit N Killing spinors. We show that there is no topological obstruction for the existence of up to 22 Killing spinors in eleven-dimensional spacetime. We investigate the symmetry superalgebras of supersymmetric backgrounds and find that their structure constants are determined by an antisymmetric matrix. The Lie subalgebra of bosonic generators is related to a real form of a symplectic group. We show that there is a one-one correspondence between certain bases of the Cartan subalgebra of $sl(32,\mathbb{R})$ and supersymmetric planar probe M-brane configurations. A supersymmetric probe configuration can involve up to 31 linearly independent planar branes and preserves one supersymmetry. The space of supersymmetric planar probe M-brane configurations is preserved by an $\mathrm{SO}(32,\mathbb{R})$ subgroup of $\mathrm{SL}(32,\mathbb{R})$.

Keywords: M-Theory, Supergravity Models.

Contents

1. Introduction

The recent classification of maximally supersymmetric solutions in ten- and eleven- dimensional supergravity theories [1] has raised the hope that the same can be achieved for solutions preserving less supersymmetry. The classification was based on the properties of the supercovariant connection, $\mathcal{D}$, for maximally supersymmetric backgrounds and the use of Plücker relations. In particular, the property that characterizes the maximally supersymmetric backgrounds is that the holonomy of the supercovariant connection is equal to one, $\mathrm{hol}(\mathcal{D}) = 1$. This implies that the curvature of the supercovariant connection vanishes, $\mathcal{R} = 0$. The vanishing of the supercovariant curvature together with the field equations is the *full set* of conditions that a background has to satisfy in order to preserve maximal supersymmetry. These conditions were solved with the use of Plücker relations and the maximal supersymmetric backgrounds were determined up to local isometries. To summarize, in [1]

- The conditions for maximal supersymmetry were derived, and
- All solutions to these conditions were determined up to local isometries.

Many supersymmetric solutions of eleven-dimensional supergravity are known and have found applications in string and M-theory. Despite this, very little has been accomplished towards a systematic investigation of supergravity solutions with less than maximal supersymmetry. There are many reasons for this. To mention some, the Riemann decomposition theorem does not apply for lorentzian manifolds, there is no Berger-type classification of holonomy groups, and there is no apparent geometric interpretation of the supercovariant connection. Apart from the classification of maximal supersymmetric backgrounds mentioned above and some other partial results [2], some progress towards understanding the backgrounds of eleven-dimensional supergravity with one Killing spinor has been made in [3]. In particular, the local form of the background which solves the Killing spinor equations, for one Killing spinor associated with a time-like Killing vector, has been constructed.

In this paper, we describe some general properties of supersymmetric M-theory backgrounds. In particular, we carry out the first part of the programme mentioned above for maximally supersymmetric solutions [1], to M-theory backgrounds with $N < 32$ supersymmetries. We present a proof that the (reduced) holonomy of the supercovariant connection $\mathcal{D}$ for backgrounds with N Killing spinors reduces to the semi-direct product of $\mathrm{SL}(32-N,\mathbb{R})$ with N-copies of $\mathbb{R}^{32-N}$,

$$\mathrm{SL}(32-N,\mathbb{R}) \ltimes \left(\oplus^N \mathbb{R}^{32-N}\right), \tag{1.1}$$

where $\mathrm{SL}(32-N,\mathbb{R})$ acts on each copy of $\mathbb{R}^{32-N}$ in the fundamental representation. This result has also been pointed out in [4] and in particular that for $N=0$ the holonomy group is $\mathrm{SL}(32,\mathbb{R})$. Therefore a simply connected M-theory background admits N Killing spinors iff the supercovariant curvature takes values in the Lie algebra of $sl(32-N,\mathbb{R}) \oplus_s (\oplus^N \mathbb{R}^{32-N})$, where $\oplus_s$ stands for the semi-direct sum. An eleven-dimensional background admits precisely N Killing spinors[1] iff

$$\mathrm{SL}(31-N,\mathbb{R}) \ltimes \left(\oplus^{N+1} \mathbb{R}^{31-N}\right) \not\supseteq \mathrm{hol}(\mathcal{D}) \subseteq \mathrm{SL}(32-N,\mathbb{R}) \ltimes \left(\oplus^N \mathbb{R}^{32-N}\right), \tag{1.2}$$

i.e. the holonomy of $\mathcal{D}$ is *contained* in $\mathrm{SL}(32-N,\mathbb{R}) \ltimes (\oplus^N \mathbb{R}^{32-N})$ but it is *not contained* in $\mathrm{SL}(31-N,\mathbb{R}) \ltimes (\oplus^{N+1} \mathbb{R}^{31-N})$. We also present the *explicit conditions* that the curvature $\mathcal{R}$ of the supercovariant connection should satisfy in order for spacetime to admit N Killing spinors. These together with the field equations of eleven-dimensional supergravity is the *full set* of conditions for a background to preserve N supersymmetries. The proof of the above results relies on the properties of the Clifford algebra in eleven-dimensions and, in particular, to its relation to the group $\mathrm{GL}(32,\mathbb{R})$. The groups $\mathrm{SL}(32,\mathbb{R})$ and $\mathrm{GL}(32,\mathbb{R})$ have recently been considered in the context of symmetries of M-theory [5, 4] and in [6]. In [6, 4] the holonomy of the supercovariant connection was also considered for spacetimes which satisfy several other conditions in addition to those of the existence of Killing spinors.

We also explore the topological properties of backgrounds with Killing spinors. We find that that there is no topological obstruction for a background to admit $N < 22$ Killing spinors. The first obstruction can occur for $N = 22$ provided that the top cohomology group of spacetime does not vanish. The obstruction is identified as the Euler class of a vector bundle. For a spacetime with topology $\mathbb{R} \times \Sigma$, an obstruction can occur at $N = 23$.

[1]We thank M. Duff and J. Liu for a discussion on this point.

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It was shown in [3] that the Killing spinor equations

$$\mathcal{D}\epsilon_i = 0\,, \qquad i = 1, \ldots, N \tag{1.3}$$

imply certain first order equations for the forms associated with the Killing spinors. Here we shall show that these first order equations are the conditions that the bispinors $\epsilon_i \otimes \epsilon_j$ are parallel with respect to supercovariant connection $\mathcal{D}$

$$\mathcal{D}(\epsilon_i \otimes \epsilon_j) = 0\,, \qquad i, j = 1, \ldots, N\,. \tag{1.4}$$

In particular, we shall show the converse, i.e. that the Killing spinor equations are equivalent to (1.4) for $i = 1$ and $j = 1, \ldots, N$.

We investigate some properties of the symmetry superalgebra of backgrounds with N Killing spinors. We show that the spinorial Lie derivative commutes with the supercovariant derivative and use this to prove closure. In addition, we show that the structure constants of the commutator

$$[T_{ij}, Q_k] = f_{ij,k}{}^l Q_l \tag{1.5}$$

can be expressed as

$$f_{ij,k}{}^p = h_{ki}\delta^p_j + h_{kj}\delta^p_i\,, \tag{1.6}$$

where h is an anti-symmetric (constant) matrix, Q are the odd generators and T are the even generators associated with the Killing vectors constructed from the Killing spinors. We also find that some of the structure constants of the commutators of two even generators are expressed in terms of the structure constants of the commutators of the even with odd generators. In particular, if the generators T_{ij} are all linearly independent and h is non-degenerate, the Lie algebra spanned by the T_{ij}s is a real form of a certain symplectic group.

We further explore the applications of the Clifford algebra in M-theory to classify all supersymmetric planar-probe M-brane configurations in $\mathbb{R}^{10,1}$. The latter are in one-one correspondence with certain bases of the Cartan subalgebra of $sl(32, \mathbb{R})$ spanned by hermitean traceless matrices. We show that the maximal number of *linearly independent* M-branes that can appear in a supersymmetric configuration is 31. This is the same as the dimension of the Cartan subalgebra of $sl(32, \mathbb{R})$. Such a configuration preserves one supersymmetry. The proof of the above statements relies on the compatibility of brane projectors. We find that $\mathrm{SL}(32, \mathbb{R})$ does not preserve the space of supersymmetric planar-probe M-brane configurations. This space is preserved by an $\mathrm{SO}(32, \mathbb{R})$ subgroup[2] of $\mathrm{SL}(32, \mathbb{R})$.

This paper is organized as follows: in section 2, we summarize the properties of the Clifford algebra in eleven-dimensions and we determine the holonomy of the supercovariant connection for backgrounds with N Killing spinors. We also find the conditions that the supercovariant curvature must satisfy in order for a background to admit N Killing spinors. In section 3, we show that there may be topological obstructions for the existence of $N \geq 22$ Killing spinors and we identify the obstruction for $N = 22$. In section 4, we investigate

[2]It is of interest to find applications of this fact in the context of symmetries of M-theory.

the symmetry superalgebras of supersymmetric backgrounds. In section 5, we classify the supersymmetric planar-probe M-brane configurations. In appendix A, we summarize some of the properties of $sl(32,\mathbb{R})$ and its relation to the Clifford algebra in eleven-dimensions. In appendix B, we define spinorial Lie derivatives along vector p-forms and discuss their applications to supersymmetric backgrounds.

2. Backgrounds with N Killing spinors

2.1 The Clifford algebra in eleven-dimensions

The Clifford algebra[3] $\mathrm{Cliff}(\mathbb{R}^{10,1})$ as a vector space is isomorphic to $\Lambda^*(\mathbb{R}^{10,1})$, $\mathrm{Cliff}(\mathbb{R}^{10,1}) = \Lambda^*(\mathbb{R}^{10,1})$, and so it has dimension 2^{11}. The Clifford algebra as an algebra is isomorphic to

$$\mathrm{Cliff}(\mathbb{R}^{10,1}) = M_{32}(\mathbb{R}) \oplus M_{32}(\mathbb{R})\,, \tag{2.1}$$

where $M_n(\mathbb{R})$ is the space of $n \times n$ matrices with real entries. The two terms in the direct sum are distinguished by the action of the element λ of the Clifford algebra associated with the volume of $\mathbb{R}^{10,1}$. This acts as $\lambda = (1,-1)$ on the two factors. The Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{10,1})$ can be written as

$$\mathrm{Cliff}(\mathbb{R}^{10,1}) = \mathrm{Cliff}^{\mathrm{even}}(\mathbb{R}^{10,1}) \oplus \mathrm{Cliff}^{\mathrm{odd}}(\mathbb{R}^{10,1})\,,$$

corresponding to the decomposition of $\Lambda^*(\mathbb{R}^{10,1})$ in terms of even- and odd-degree forms. It is known that

$$\mathrm{Cliff}^{\mathrm{even}}(\mathbb{R}^{10,1}) = M_{32}(\mathbb{R}) \tag{2.2}$$

and so $\mathrm{Spin}(10,1) \subset \mathrm{Cliff}^{\mathrm{even}}(\mathbb{R}^{10,1}) = M_{32}(\mathbb{R})$. The Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{10,1})$ has two irreducible (pinor) representations, both of dimension 32. These are given by the standard action of $M_{32}(\mathbb{R})$ on $\mathbb{R}^{32}$, for each factor in the decomposition (2.1). The even part of the Clifford algebra has a unique irreducible representation given by the standard action of $M_{32}(\mathbb{R})$ on $\mathbb{R}^{32}$, as it can be seen from (2.2). This restricts to an irreducible real-spinor (Majorana) representation Δ on $\mathrm{Spin}(10,1)$. The spinor representation Δ is equipped with a (real) non-degenerate skew-symmetric $\mathrm{Spin}(10,1)$-invariant inner product C, the charge conjugation matrix.[4] Therefore we have

$$\mathrm{Spin}(10,1) \subset Sp(32,\mathbb{R}) \subset \mathrm{SL}(32,\mathbb{R}) \subset \mathrm{GL}(32,\mathbb{R})\,. \tag{2.3}$$

The product of two spinor representations can be decomposed as

$$\Delta \otimes \Delta = \sum_{n=0}^{5} \Lambda^n(\mathbb{R}^{10,1})\,.$$

This is equivalent to the fact that a bi-spinor $\bar{\chi} \otimes \psi$ can be written as

$$\bar{\chi} \otimes \psi = \frac{1}{32}\left\{ (\bar{\chi}\psi) 1_{32} + \Gamma^A (\bar{\chi}\Gamma_A \psi) + \cdots + \frac{1}{5!}\Gamma^{A_1\dots A_5}\,(\bar{\chi}\Gamma_{A_1\dots A_5}\psi) \right\}, \tag{2.4}$$

[3] All these are well-known results and are summarised e.g. in [7].

[4] The inner product C can be used to identify Δ and its dual.

where $\bar{\chi} = \chi^t C^{-1}$ and $\{\Gamma^A; A = 0, \dots 10\}$ is a basis of gamma matrices. This basis can be chosen such that Γ^0 is anti-hermitean and the rest are hermitean. Since the representation is real, all the gamma matrices can be taken to be real and so Γ^0 is skew-symmetric while the rest are symmetric matrices. All skew products of gamma matrices are traceless. So taking the trace in (2.4), we get an identity. Since a bi-spinor can be viewed as an element in $M_{32}(\mathbb{R})$, the right-hand-side of (2.4) takes values in $M_{32}(\mathbb{R})$.

To relate the Killing spinor equations to the parallel transport equations of forms constructed from bi-spinors, we need a relation between spinors and forms. For this, we choose a spinor $\epsilon \neq 0$. Then setting $\chi = \epsilon$ in (2.4), we can define a map i_ϵ from Δ into $\Lambda^*(\mathbb{R}^{10,1})$ by setting

$$i_\epsilon(\psi) = \bar{\epsilon} \otimes \psi \,, \tag{2.5}$$

i.e. $(i_\epsilon(\psi))^\alpha{}_\beta = (\epsilon^t C^{-1})^\alpha \psi_\beta$. We now prove that this map is $1-1$. Since the map i_ϵ is linear, we have to show that its kernel vanishes. Indeed, suppose that there is a $\psi \neq 0$ in the kernel of the linear map. This implies that $\chi_\alpha \epsilon_\beta = 0$ for all α, β. However since neither ψ nor ϵ vanish, they must have at least one non-vanishing component each. Thus there is at least a pair (α, β) such that $\chi_\alpha \epsilon_\beta \neq 0$. Therefore the Kernel contains only the zero element and the map is $1 - 1$. This result implies that if there is a distinguished non-vanishing spinor, we can describe any other spinor in terms of forms which in the case of eleven-dimensional supergravity have rank from zero to five. Therefore given a non-vanishing spinor ϵ, Δ can be viewed as a subspace of $\sum_{n=0}^{5} \oplus \Lambda^n(\mathbb{R}^{10,1})$.

2.2 The holonomy of the supercovariant connection and N Killing spinors

The spinor bundle S can be viewed as an associated vector bundle of principal bundles for any of the groups in (2.3), i.e. $S = P(G) \times_\rho \Delta$ for $G = \mathrm{Spin}(10,1), Sp(32,\mathbb{R}), \mathrm{SL}(32,\mathbb{R})$ or $\mathrm{GL}(32,\mathbb{R})$. The representation ρ is the standard representation of $\mathrm{GL}(32,\mathbb{R})$ on $\Delta = \mathbb{R}^{32}$ restricted on the subgroups G.

It turns out that for the investigation of Killing spinors in the context of eleven-dimensional supergravity, it is most convenient to view S as an associated bundle of $\mathrm{SL}(32,\mathbb{R})$. This is because the (reduced) holonomy of the supercovariant derivative $\mathcal{D}$ is contained in $\mathrm{SL}(32,\mathbb{R})$, $\mathrm{hol}(\mathcal{D}) \subseteq \mathrm{SL}(32,\mathbb{R})$. The supercovariant derivative of eleven-dimensional supergravity [8] is

$$\mathcal{D}_M \epsilon = \nabla_M \epsilon + \Omega_M \epsilon \,, \tag{2.6}$$

where

$$\Omega_M = -\frac{1}{288} \left(\Gamma_M{}^{PQRS} F_{PQRS} - 8 F_{MPQR} \Gamma^{PQR} \right) , \tag{2.7}$$

∇_M is the spin covariant derivative induced from the Levi-Civita connection and M, N, P, Q, R, $S = 0, \dots, 10$ are spacetime indices. The curvature of the supercovariant derivative, $\mathcal{R}$, is defined as,

$$\mathcal{R}_{MN} = [\mathcal{D}_M, \mathcal{D}_N] \tag{2.8}$$

and it can be expanded in a basis of gamma matrices as

$$\mathcal{R}_{MN} = \sum_{n=1}^{5} \phi_{MN}{}^{A_1 \dots A_n} \Gamma_{A_1 \dots A_n} \,, \tag{2.9}$$

where $A_1, A_2, \ldots, A_n = 0, 1, 2, \ldots, 10$ are frame indices of eleven-dimensional spacetime. The coefficients ϕ are functions of the bosonic fields of eleven-dimensional supergravity (g, F). An explicit expression for these coefficients can be found in [9, 1]. The Lie algebra of the holonomy group of a connection D is determined by the span of the values of the curvature $R_D(X,Y)$ evaluated on any two vector fields X, Y. The supercovariant curvature takes values in $\Delta \otimes \Delta$ but it *does not* contain any term which is *zeroth order* in gamma matrices, i.e. proportional to 1_{32}. Thus the trace

$$\mathrm{tr}\left(\mathcal{R}(X,Y)\right) = 0$$

on the spinor indices vanishes. Therefore $\mathcal{R}(X,Y)$ takes values in the subset $M^0_{32}(\mathbb{R}) \subset M_{32}(\mathbb{R})$ of 32×32 real matrices which have vanishing trace. The latter can be identified with the Lie algebra $sl(32,\mathbb{R}) = M^0_{32}(\mathbb{R})$ of $\mathrm{SL}(32,\mathbb{R})$. Thus the *reduced holonomy group* of the supercovariant derivative is contained in $\mathrm{SL}(32,\mathbb{R})$, $\mathrm{hol}(\mathcal{D}) \subseteq \mathrm{SL}(32,\mathbb{R})$. For a different proof of this see [4]. In what follows, we shall consider $S = P(\mathrm{SL}(32,\mathbb{R})) \times_\rho \Delta$. The supercovariant connection and supercovariant curvature are not hermitean elements of the Clifford algebra.

The existence of parallel (Killing) spinors with respect to the supercovariant derivative $\mathcal{D}$, i.e. spinors ϵ such that

$$\mathcal{D}_M \epsilon = 0\,,$$

implies

$$\mathcal{R}_{MN}\epsilon = 0$$

and the holonomy group reduces to a subgroup of $\mathrm{SL}(32,\mathbb{R})$. We shall identify the subgroup of $\mathrm{SL}(32,\mathbb{R})$ which allows the presence of N Killing spinors. This is equivalent to finding the subgroup of $\mathrm{SL}(32,\mathbb{R})$ which is the stability subgroup of N spinors in $\Delta = \mathbb{R}^{32}$. The reduction of the holonomy group of the Levi-Civita connection due to the existence of one Killing spinor has been investigated in [10, 11]. The group $\mathrm{SL}(32,\mathbb{R})$ acting with the standard representation on $\Delta = \mathbb{R}^{32}$ has two orbits. One is the origin $\{0\}$ of $\mathbb{R}^{32}$ and the other is $\mathbb{R}^{32} - \{0\}$. The stability subgroup of a non-vanishing spinor is

$$\mathrm{SL}(31,\mathbb{R}) \ltimes \mathbb{R}^{31}\,.$$

In matrix notation, any $A \in \mathrm{SL}(31,\mathbb{R}) \ltimes \mathbb{R}^{31}$ can be written as

$$A = \begin{pmatrix} 1 & u^t \\ 0 & B \end{pmatrix}\,, \tag{2.10}$$

where $u \in \mathbb{R}^{31}$ and $B \in \mathrm{SL}(30,\mathbb{R})$. Using this, it is easy to show that $\mathrm{SL}(32,\mathbb{R})/\,\mathrm{SL}(31,\mathbb{R}) \ltimes \mathbb{R}^{31} = \mathbb{R}^{32} - \{0\}$. Next suppose that there are two linearly independent Killing spinors ϵ_1 and ϵ_2. Without loss of generality, we can choose the two spinors to be

$$\epsilon_1 = e_1 \qquad \epsilon_2 = \sum_{i=1}^{32} v^i e_i\,, \tag{2.11}$$

where $\{e_i; i = 1, 2, \ldots, 32\}$ is the standard basis in $\mathbb{R}^{32}$. Since ϵ_1 and ϵ_2 are linearly independent, one of the components $v_2, \ldots, v_{32}$ of ϵ_2 must be non-vanishing. Since $\mathrm{SL}(31,\mathbb{R})$

acts transitively on $\mathbb{R}^{31} - \{0\}$, we can use the stability subgroup of ϵ_1 to set $v_3 = v_4 = \ldots = v_{32} = 0$. It is then straightforward to see that the stability subgroup of both ϵ_1 and ϵ_2 is $\mathrm{SL}(30,\mathbb{R}) \ltimes (\mathbb{R}^{30} \oplus \mathbb{R}^{30})$. The group $\mathrm{SL}(30,\mathbb{R})$ acts with the fundamental representation on both vector spaces in the direct sum. In matrix representation, any $A \in \mathrm{SL}(30,\mathbb{R}) \ltimes (\mathbb{R}^{30} \oplus \mathbb{R}^{30})$ can be written as

$$A = \begin{pmatrix} 1 & 0 & u_1^t \\ 0 & 1 & u_2^t \\ 0 & 0 & B \end{pmatrix}, \tag{2.12}$$

where $u_1, u_2 \in \mathbb{R}^{30}$ and $B \in \mathrm{SL}(30,\mathbb{R})$. The orbits of $\mathrm{SL}(31,\mathbb{R}) \ltimes \mathbb{R}^{31}$ in $\mathbb{R}^{32}$ are either the points of the line $\mathbb{R}(e_1)$ along the direction of the e_1 axis, or $\mathbb{R}^{32}$ with the line along e_1 removed. I.e.

$$\begin{aligned} \mathcal{O}_x &= \{x \in \mathbb{R}(e_1)\} \\ \mathcal{O}' &= \mathbb{R}^{32} - \mathbb{R}(e_1)\,. \end{aligned} \tag{2.13}$$

The stability subgroup of a spinor in $\mathbb{R}^{32} - \mathbb{R}(e_1)$ is $\mathrm{SL}(30,\mathbb{R}) \ltimes (\mathbb{R}^{30} \oplus \mathbb{R}^{30})$ and so

$$\mathrm{SL}(31,\mathbb{R}) \ltimes \mathbb{R}^{31} / \mathrm{SL}(30,\mathbb{R}) \ltimes (\mathbb{R}^{30} \oplus \mathbb{R}^{30}) = \mathbb{R}^{32} - \mathbb{R}(e_1)\,. \tag{2.14}$$

Continuing in the same fashion, the stability subgroup of N linearly independent Killing spinors is

$$\mathrm{SL}(32-N,\mathbb{R}) \ltimes (\oplus^N \mathbb{R}^{32-N})\,. \tag{2.15}$$

In matrix representation, an element $A \in \mathrm{SL}(32-N,\mathbb{R}) \ltimes (\oplus^N \mathbb{R}^{32-N})$ can be written as

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & u_1^t \\ 0 & 1 & 0 & \ldots & 0 & 0 & u_2^t \\ & & \vdots & & \vdots & & \\ 0 & 0 & 0 & \ldots & 0 & 1 & u_k^t \\ 0 & 0 & 0 & \ldots & 0 & 0 & B \end{pmatrix}, \tag{2.16}$$

where $u_1, \ldots, u_N \in \mathbb{R}^{32-N}$ and $B \in \mathrm{SL}(32-N,\mathbb{R})$. The orbits of $\mathrm{SL}(33-N,\mathbb{R}) \ltimes (\oplus^{N-1}\mathbb{R}^{33-N})$ in $\mathbb{R}^{32}$ are as follows:

$$\begin{aligned} \mathcal{O}_x &= \{x \in \mathbb{R}(e_1, \ldots, e_{k-1})\} \\ \mathcal{O}' &= \mathbb{R}^{32} - \mathbb{R}(e_1, \ldots, e_{k-1})\,. \end{aligned} \tag{2.17}$$

The stability subgroup of an element in $\mathbb{R}^{32} - \mathbb{R}(e_1, \ldots, e_{N-1})$ is $\mathrm{SL}(32-N,\mathbb{R}) \ltimes (\oplus^N \mathbb{R}^{32-N})$. Thus we have

$$\mathrm{SL}(33-N,\mathbb{R}) \ltimes (\oplus^{N-1}\mathbb{R}^{33-N}) / \mathrm{SL}(32-N,\mathbb{R}) \ltimes (\oplus^N \mathbb{R}^{32-N}) = \mathbb{R}^{32} - \mathbb{R}(e_1, \ldots, e_{N-1})\,. \tag{2.18}$$

To summarize, we have shown that the holonomy of the supercovariant connection associated with a spacetime which admits N Killing spinors is a subgroup of SL(32 −

$N, \mathbb{R}) \ltimes (\oplus^N \mathbb{R}^{32-N})$. This result has also been stated in [4], here we have given the proof in detail. For the spacetime to admit precisely N Killing spinors the holonomy of the supercovariant connection must satisfy

$$\mathrm{SL}(31-N,\mathbb{R}) \ltimes (\oplus^{N+1}\mathbb{R}^{31-N}) \nsupseteq \mathrm{hol}(\mathcal{D}) \subseteq \mathrm{SL}(32-N,\mathbb{R}) \ltimes (\oplus^N \mathbb{R}^{32-N}) .$$

It is clear from the discussion in this section that at every point of spacetime, $\mathrm{SL}(32,\mathbb{R})$ acts transitively on the coset $\mathrm{SL}(32,\mathbb{R})/\,\mathrm{SL}(32-N,\mathbb{R}) \ltimes (\oplus^N\mathbb{R}^{32-N})$. The latter can be thought of as the space of inequivalent configurations with N Killing spinors.[5] However as we shall see $\mathrm{SL}(32,\mathbb{R})$ does not preserve the space of supersymmetric planar probe M-brane configurations. It would be of interest to see whether this difference of behaviour between the $\mathrm{SL}(32,\mathbb{R})$ and $\mathrm{SO}(32,\mathbb{R})$ groups has applications in the context of symmetries of M-theory. It is the latter group that acts on the M-brane charges.

2.3 Supercovariant curvature and N Killing spinors

As we have seen the necessary and sufficient condition for the existence of N Killing spinors is the reduction of the holonomy of the supercovariant derivative $\mathcal{D}$ from $\mathrm{SL}(32,\mathbb{R})$ to the $\mathrm{SL}(32-N,\mathbb{R}) \ltimes (\oplus^N\mathbb{R}^{32-N})$ subgroup. However, this may be seen as rather implicit so we shall express this condition in terms of relations for the metric g and the four-form field strength F of eleven-dimensional supergravity. Let ϵ be a Killing spinor, then we have seen that ϵ is a eigenvector of $\mathcal{R}_{MN}$ with zero eigenvalue

$$\mathcal{R}_{MN}\epsilon = 0 . \tag{2.19}$$

For simply connected spacetimes (2.19) is also a sufficient condition.[6] Since $M^0_{32} = sl(32,\mathbb{R})$, we can express the supercovariant curvature in terms of a basis $\{\mathrm{m}_{ab}; a,b = 1,\ldots,32\}$ basis of $gl(32,\mathbb{R})$. I.e.

$$\mathcal{R}_{MN} = \sum_{n=1}^{5} \phi_{MN}{}^{A_1A_2\ldots A_n}\Gamma_{A_1A_2\ldots A_n} = \sum_{n=1}^{5} \phi^{A_1A_2\ldots A_n}_{MN} X^{ab}_{A_1\ldots A_n}\mathrm{m}_{ab} = \mathcal{R}^{ab}_{MN}\mathrm{m}_{ab} , \tag{2.20}$$

where X is the transformation from the basis of gamma matrices basis to the basis of $gl(32,\mathbb{R})$, see appendix A. We can adapt an $\mathrm{SL}(32,\mathbb{R})$ moving frame in spacetime, such that ϵ is along the e_1 axis. Using the basis m_{ab} defined in appendix A, the conditions on the supercovariant curvature for a spacetime to admit one Killing spinor are

$$\mathcal{R}^{a1}_{MN} = 0 , \qquad a = 1, \ldots, 32 . \tag{2.21}$$

Similarly, the conditions for spacetime to admit N Killing spinors are

$$\mathcal{R}^{ab}_{MN} = 0 , \qquad a = 1, \ldots, 32 , \qquad b = 1, \ldots, N . \tag{2.22}$$

[5]This is similar to the statement that $\mathrm{GL}(2n,\mathbb{R})/\,\mathrm{GL}(n,\mathbb{C})$ parameterizes the space of complex structures at a point.

[6]We assume that the spacetime is simply connected. If it is not, we take the universal cover.

These together with the field equations of eleven-dimensional supergravity is the *full set* of conditions that a spacetime should satisfy in order for it to admit N Killing spinors. This extends the first part of the programme for the classification of maximal supersymmetric solutions [1] to backgrounds that preserve less than maximal supersymmetry. The conditions we have derived, although explicit, are rather involved and it remains to be seen whether the second part of the programme of [1] can be carried over to the $N < 32$ case as well.

3. Topological Aspects of Spacetimes with N Killing spinors

There may be topological conditions for the existence of N parallel spinors in spacetime. Since the holonomy of the supercovariant connection reduces, the structure group of the spinor bundle reduces as well to a subgroup of the holonomy group (see e.g. [12]). In particular, the structure group of S reduces from $\mathrm{SL}(32,\mathbb{R})$ to the subgroup $\mathrm{SL}(32-N,\mathbb{R})\ltimes(\oplus^N\mathbb{R}^{32-N})$. The topological conditions for the existence of N Killing spinors are identified with the obstructions to the above reduction of the structure group. It is known that there are obstructions in the reduction of the structure group of the spin bundle due to the existence of a no-where vanishing spinor on two-, four- and eight-dimensional manifolds, see [13, 14] and references therein. For manifolds with dimension more than eight there is no topological obstruction because the rank of the spin bundle is much greater than the dimension of the manifold. In particular in eleven-dimensions there is no obstruction for the existence of a no-where vanishing spinor. However, the reduction of the structure group from $\mathrm{SL}(32,\mathbb{R})$ to $\mathrm{SL}(32-N,\mathbb{R})\ltimes(\oplus^N\mathbb{R}^{32-N})$ imposes some topological conditions on spacetime for large enough N, as we now show.

It is well-known that the structure group of S reduces from $\mathrm{SL}(32,\mathbb{R})$ to $\mathrm{SL}(32-N,\mathbb{R})\ltimes(\oplus^N\mathbb{R}^{32-N})$ iff the associated bundle $P(\mathrm{SL}(32,\mathbb{R}))\times_{\bar\rho} Z$, with fibre Z the coset space

$$Z=\mathrm{SL}(32,\mathbb{R})/\,\mathrm{SL}(32-N,\mathbb{R})\ltimes(\oplus^N\mathbb{R}^{32-N})\,, \tag{3.1}$$

admits a global section. In (3.1), $\bar\rho$ is the left action of $\mathrm{SL}(32,\mathbb{R})$ on Z. There may be obstructions to the existence of such a global section. These obstructions are elements in the cohomology groups of spacetime.[7] The first obstruction class, called primary, lies in the first non-vanishing cohomology group $H^n(M,\pi_{n-1}(Z))$. If the primary obstruction vanishes, there may be secondary obstructions which lie in higher cohomology groups (see e.g. [15]). In our case, in order to find the cohomology group of spacetime in which the primary obstruction lies, observe that the coset space Z is homotopic to the Stiefel manifold $V_{32,N}=\mathrm{SO}(32,\mathbb{R})/\,\mathrm{SO}(32-N,\mathbb{R})$. This can be seen by observing that $\mathrm{SL}(K,\mathbb{R})$ contracts to its maximal compact subgroup $\mathrm{SO}(K)$. The Stiefel manifold $V(K,N)=\mathrm{SO}(K)/\,\mathrm{SO}(K-N)$ is $K-N-1$ connected. This means that

$$\pi_r(V(K,N))=0\qquad r=0,\dots,K-N-1\,.$$

[7] We assume that the spacetime is homotopic to a CW complex.

In addition,

$$
\begin{aligned}
\pi_r(V(r+N,N)) &= \mathbb{Z} \qquad r \in 2\mathbb{Z} \\
\pi_r(V(r+N,N)) &= \mathbb{Z}_2 \qquad r \in 2\mathbb{Z}+1\,, \quad N \neq 1 \\
\pi_r(V(r+1,1) &= \mathbb{Z} \qquad r \in \mathbb{Z}\,.
\end{aligned} \tag{3.2}
$$

Because $V_{32,N}$ is $31-N$ connected, the primary obstruction lies in the cohomology group

$$
H^{33-N}(M, \pi_{32-N}(V_{32,N}))\,. \tag{3.3}
$$

If spacetime is homotopic to a point, the primary and all the secondary obstructions vanish and so there are no topological conditions for the existence of any number of Killing spinors. Obstructions may arise when the spacetime is topologically non-trivial.

For one Killing spinor, $N=1$, in eleven-dimensional spacetime, (3.3) always vanishes. So there is no primary obstruction. There are no secondary obstructions either because they lie in cohomology groups of higher degree. So there is no topological restriction on spacetime for a single Killing spinor to exist. The same is true for $N<22$. For example there is no topological obstruction for the existence of $N=16$ Killing spinors in eleven-dimensional spacetime.

If $N=22$ and $H^{11}(M)\neq 0$, there may be a non-vanishing primary obstruction which lies in $H^{11}(M,\pi_{10}(V_{32,22}))=H^{11}(M,\mathbb{Z})$. To identify this obstruction, let us assume that we have N Killing spinors, and let Δ_N be the $\mathrm{SL}(32-N,\mathbb{R})\ltimes(\oplus^N\mathbb{R}^{32-N})$-invariant subspace in Δ. We have that

$$
0\to\Delta_N\to\Delta\to\Delta/\Delta_N\to 0\,. \tag{3.4}
$$

Observe that this short exact sequence does not split, i.e. Δ cannot be written as a direct sum of Δ_N and another space in such way that the sum is preserved by the $\mathrm{SL}(32-N,\mathbb{R})\ltimes(\oplus^N\mathbb{R}^{32-N})$ group action. Using this exact sequence of vector spaces, we can define the bundles $S_N=M\times\Delta_N$, S and $S_{\Delta/\Delta_N}=P(\mathrm{SL}(32-N,\mathbb{R})\ltimes(\oplus^N\mathbb{R}^{32-N}))\times_\rho\Delta/\Delta_N$, where S is the spin bundle, ρ is the factor representation of $\mathrm{SL}(32-N,\mathbb{R})\ltimes(\oplus^N\mathbb{R}^{32-N})$ on Δ/Δ_N and S_N is the bundle of Killing spinors. Since there is no topological obstruction for the existence of $N=21$ Killing spinors, we can assume that the spacetime admits $N=21$ Killing spinors and so we can define the bundle $S_{\Delta/\Delta_{21}}$. The obstruction for the existence of $N=22$ Killing spinors is the Euler class of the bundle $S_{\Delta/\Delta_{21}}$. (Observe this bundle has the same rank as the dimension of spacetime.) If this Euler class vanishes, there are no further secondary obstructions.

4. Killing spinors and forms

The supercovariant derivative $\mathcal{D}$, as any other covariant derivative of the spin bundle S, can be extended to a covariant derivative of $S\otimes S$ as $\mathcal{D}\otimes 1+1\otimes\mathcal{D}$. But $S\otimes S$ can be thought of as a sub-bundle of $\Lambda^*(M)$. So $\mathcal{D}$ can be extended to a covariant derivative on the space of forms. In particular, we have

$$\begin{aligned}\mathcal{D}_M(\chi_\alpha\psi_\beta) &= \mathcal{D}_M\chi_\alpha\psi_\beta + \chi_\alpha\mathcal{D}_M\psi_\beta \\ &= \frac{1}{32}\Bigg[-C_{\alpha\beta}\mathcal{D}_M\bar{\chi}\psi + \Gamma^N_{\alpha\beta}\mathcal{D}_M\bar{\chi}\Gamma_N\psi + \cdots + \frac{1}{5!}\Gamma^{N_1\dots N_5}_{\alpha\beta}\mathcal{D}_M\bar{\chi}\Gamma_{N_1\dots N_5}\psi - \\ &\quad - C_{\alpha\beta}\bar{\chi}\mathcal{D}_M\psi + \Gamma^N_{\alpha\beta}\bar{\chi}\Gamma_N\mathcal{D}_M\psi + \cdots + \frac{1}{5!}\Gamma^{N_1\dots N_5}_{\alpha\beta}\bar{\chi}\Gamma_{N_1\dots N_5}\mathcal{D}_M\psi\Bigg]\,. \end{aligned} \tag{4.1}$$

Using (2.6), we can rewrite the above expression as

$$\begin{aligned}\mathcal{D}_M(\chi_\alpha\psi_\beta) &= \frac{1}{32}\Bigg[-C_{\alpha\beta}\nabla_M(\bar{\chi}\psi) + \Gamma^N_{\alpha\beta}\nabla(\bar{\chi}\Gamma_N\psi) + \cdots + \frac{1}{5!}\Gamma^{N_1\dots N_5}_{\alpha\beta}\nabla_M(\chi\Gamma_{N_1\dots N_5}\psi) - \\ &\quad - C_{\alpha\beta}\bar{\chi}(\bar{\Omega}_M + \Omega_M)\psi + \Gamma^N_{\alpha\beta}\bar{\chi}(\bar{\Omega}_M\Gamma_N + \Gamma_N\Omega_M)\psi + \cdots + \\ &\quad + \frac{1}{5!}\Gamma^{N_1\dots N_5}_{\alpha\beta}\bar{\chi}(\bar{\Omega}_M\Gamma_{N_1\dots N_5} + \Gamma_{N_1\dots N_5}\Omega_M)\psi\Bigg]\,. \end{aligned} \tag{4.2}$$

The supercovariant derivative of the k-form $\bar{\chi}\Gamma_{N_1\dots N_k}\psi$ associated with the bi-spinor $\chi\otimes\psi$ is

$$\mathcal{D}_M(\bar{\chi}\Gamma_{N_1\dots N_k}\psi) = \nabla_M(\bar{\chi}\Gamma_{N_1\dots N_k}\psi) + \bar{\chi}(\bar{\Omega}_M\Gamma_{N_1\dots N_k} + \Gamma_{N_1\dots N_k}\Omega_M)\psi\,. \tag{4.3}$$

The second part of the right-hand-side of (4.3) can be rewritten in terms of forms associated with the bi-spinor $\chi\otimes\psi$ contracted with F. For a generic four-form field strength F, $\mathcal{D}$ does not preserve the degree of the form that it acts on. In other words, when the connection is evaluated on a one-form, the term involving Ω_M mixes the forms of various degrees.

Suppose now that $\chi = \epsilon_i$ and $\psi = \epsilon_j$ are Killing spinors, so that $\mathcal{D}_M(\epsilon_i\otimes\epsilon_j) = 0$. This implies that the forms associated with the bi-spinors $\epsilon_i\otimes\epsilon_j$ are parallel with respect to the superconnection $\mathcal{D}$. Let us denote the forms associated with the bi-spinor $\epsilon_i\otimes\epsilon_j$ collectively by τ_{ij}. Then we have that

$$\mathcal{D}_M\tau_{ij} = 0\,, \qquad i,j = 1,\dots,N\,. \tag{4.4}$$

These conditions are explicitly given in [3] after adjusting for conventions ($F\to -F$).

Here, we shall show that only N of the conditions (4.4) are independent, as many as the number of Killing spinors. In addition, we shall show that these N independent conditions are equivalent to the Killing spinor equations. To show this recall that if there is a non-vanishing spinor ϵ, one can construct an inclusion of Δ in $\Lambda^*(\mathbb{R}^{10,1})$ as in (2.5). Setting $\epsilon=\epsilon_1$, the independent conditions are equivalent to $\epsilon_1\otimes\epsilon_i$ for $i=1,\dots,N$ being parallel with respect to $\mathcal{D}$. Therefore the independent conditions are

$$\mathcal{D}_M\tau_{1i} = 0\,, \qquad i = 1,\dots,N\,. \tag{4.5}$$

Conversely, if (4.5) is satisfied, then $\mathcal{D}_M\epsilon_i = 0$. To show this consider the spinor $\epsilon = \epsilon_1 \neq 0$ and suppose that $\mathcal{D}\epsilon\neq 0$ but $\mathcal{D}\tau_{11}=0$. In this case we have

$$\mathcal{D}\tau_{11} = 0 = \mathcal{D}\epsilon\otimes\epsilon + \epsilon\otimes\mathcal{D}\epsilon\,. \tag{4.6}$$

It is easy to see that if there is a component of $\mathcal{D}\epsilon$ which does vanish, then $\epsilon = 0$ which is a contradiction. Next we take

$$0 = \mathcal{D}\tau_{1i} = \mathcal{D}(\epsilon \otimes \epsilon_i) = \epsilon \otimes \mathcal{D}\epsilon_i \,, \tag{4.7}$$

which implies that $\mathcal{D}\epsilon_i = 0$. This completes the proof that the Killing spinor equations are equivalent to (4.5).

5. Symmetry superalgebra

The bosonic symmetries of a supersymmetric background of a supergravity theory are generated by Killing vectors which in addition leave the remaining form field-strengths invariant. There are two types of such Killing vectors depending on whether or not they can be constructed from Killing spinors. The Killing vectors

$$K_{ij} = K_{ji} = \bar{\epsilon}_i \Gamma^M \epsilon_j \partial_M \,, \qquad i,j = 1,\ldots,N \tag{5.1}$$

associated with the Killing spinors ϵ_i, ϵ_j are not all independent. The maximal number of linearly independent Killing vectors K_{ij} is $\frac{1}{2}N(N+1)$. For $N > 11$, this number is larger than the maximal number of linearly independent Killing vectors allowed on eleven-dimensional spacetime. In most known supersymmetric M-theory backgrounds, the number of K_{ij} Killing vectors is further restricted. For example pp-wave backgrounds associated with multi-centre harmonic functions admit a single null Killing vector despite the fact that they preserve sixteen supersymmetries, $N = 16$.

Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be the symmetry superalgebra of an M-theory background — $\mathcal{G}_0$ and $\mathcal{G}_1$ are the even and odd parts, respectively. In addition, we introduce a basis $\{Q_i : i = 1,\ldots,N\}$ of $\mathcal{G}_1$. The even part $\mathcal{G}_0$ is spanned by $\{T_{ij} : i,j = 1,\ldots,N\}$ associated with Killing vectors (5.1) and by $\{T_a : a = 1,\ldots,I\}$ which are the even generators associated with the rest of the Killing vectors of the M-theory background. The main aim of this section is to show that the symmetry superalgebra of a supersymmetric M-theory background can be written as

$$\begin{aligned}
[T_{ij}, Q_k] &= f_{ij,k}{}^l Q_l \\
[T_a, Q_i] &= f_{ai}{}^l Q_l \\
[T_{ij}, T_{mn}] &= f_{ij,mn}{}^{kl} T_{kl} \\
[T_a, T_{ij}] &= f_{ai}{}^k T_{kj} + f_{aj}{}^k T_{ik} \\
[T_a, T_b] &= f_{ab}{}^c T_c + f_{ab}{}^{ij} T_{ij} \\
\{Q_i, Q_j\} &= T_{ij} \,.
\end{aligned} \tag{5.2}$$

The f's above are structure constants obeying

$$f_{ij,mn}{}^{kl} = f_{ij,m}{}^{(k} \delta^{l)}_n + f_{ij,n}{}^{(k} \delta^{l)}_m \tag{5.3}$$

and

$$f_{ij,k}{}^{p} = h_{ki}\delta^p_j + h_{kj}\delta^p_i\,, \tag{5.4}$$

where h_{ij} is a constant antisymmetric second-rank tensor, $h_{ij} = -h_{ji}$. Note that (5.3) and (5.4) imply that the structures constants $f_{ij,mn}{}^{kl}$ are antisymmetric under $(m,n) \leftrightarrow (i,j)$. In (5.2) we have treated all the T_{ij} generators as independent. However as we have mentioned this is not always the case because the associated Killing vectors K_{ij} may be linearly dependent. If the T_{ij}s are linearly dependent, then the constant anti-symmetric matrix h should be appropriately restricted. The generators T_{ij} span a subalgebra of $\mathcal{G}_0$ which we denote with $\mathcal{G}^s_0$

We shall first show closure. It is well known that the algebra of Killing vectors on a manifold closes. This implies the closure for all the even generators. The commutator of even with odd generators can be computed from the spinorial Lie derivative of the Killing spinors along the direction of the Killing vector fields [16, 17]. To show the closure of the commutator of even with odd generators, we have to show that the spinorial Lie derivative of a Killing spinor with respect to any Killing vector which is a symmetry of the background is again a Killing spinor. Indeed, let K be any vector which satisfies

$$\mathcal{L}_K g = 0 \qquad \mathcal{L}_K F = 0 \tag{5.5}$$

and ϵ be a Killing spinor. The spinorial Lie derivative of ϵ along K is

$$\mathcal{L}_K\epsilon = K^M\nabla_M\epsilon + \frac{1}{4}\nabla_M K_N\Gamma^{MN}\epsilon\,. \tag{5.6}$$

Using (5.5) and the properties of the spinorial Lie derivative, we find that

$$[\mathcal{L}_K, \mathcal{D}] = 0\,. \tag{5.7}$$

So if $\mathcal{D}\epsilon = 0$, then $\mathcal{D}\mathcal{L}_K\epsilon = 0$ and the statement is shown.

Closure of the commutator of even with odd generators implies that there are (structure) constants f such that

$$\mathcal{L}_{K_{ij}}\epsilon_k = f_{ij,k}{}^{l}\epsilon_l\,. \tag{5.8}$$

To show that f is constant, it suffices to take the supercovariant derivative on both sides of (5.8). Using (5.8), we can determine the structure constants associated with the Lie bracket of the Killing vectors,

$$\begin{aligned}[K_{ij}, K_{mn}] &= \mathcal{L}_{K_{ij}}(\bar\epsilon_m\Gamma\epsilon_n) \\ &= f_{ij,m}{}^{\ell}K_{\ell n} + f^{ij,n\ell}K_{\ell m}\,. \end{aligned} \tag{5.9}$$

This establishes (5.3) and the commutator of the even generators associated with Killing vector fields constructed from Killing spinors.

Closure also implies that there are constants such that

$$\mathcal{L}_{K_a}\epsilon_i = f_{ai}{}^{j}\epsilon_j\,. \tag{5.10}$$

One can then easily determine the structure constants of the commutator $[T_a, T_{ij}]$ from those above. We have thus shown that most of the structure constants are determined from the brackets of the even generators with the odd generators.

It may seem natural to take $f_{ab}{}^{ij} = 0$ in (5.2)so that the even $\{T_a\}$ generators act on the remaining algebra as external automorphisms. In many known examples, like the maximal supersymmetric plane-waves ([18]–[20]), it holds that $f_{ab}{}^{ij} = 0$. However, it is not apparent that these structure constants will vanish in general.

The commutator of the odd with odd generators can be derived from the squaring of two Killing spinors. Therefore we have

$$\{Q_i, Q_j\} = T_{ij} \,. \tag{5.11}$$

(See appendix B for a further discussion.) In order to show (5.4), we first note that

$$f_{(ij,k)}{}^{l} = 0 \,. \tag{5.12}$$

Indeed, let us examine (5.8) more closely. The Lie derivative of a spinor transforms as the symmetrized tensor product of two spinors tensored with a four-form and a spinor,

$$\mathcal{L}_{K_{ij}}\epsilon_k \sim (00001)^{2\otimes_s} \otimes (00010) \otimes (00001) \sim 10(00001) \oplus \cdots \,, \tag{5.13}$$

where we have noted that there are ten spinors in the decomposition. These can be given explicitly,

$$\begin{aligned}
S^{(1)}_{ij,k} &:= \frac{1}{3!}(\Gamma_{N_1N_2N_3}\epsilon_k)(K_{ij})_{N_4}F^{N_1\ldots N_4}\\
S^{(2)}_{ij,k} &:= \frac{1}{4!}(\Gamma_{N_1\ldots N_5}\epsilon_k)(K_{ij})^{N_1}F^{N_2\ldots N_5}\\
S^{(3)}_{ij,k} &:= \frac{1}{4}(\Gamma_{N_1N_2}\epsilon_k)(\Omega_{ij})_{N_3N_4}F^{N_1\ldots N_4}\\
S^{(4)}_{ij,k} &:= \frac{1}{4!}(\Gamma_{N_1\ldots N_4}\epsilon_k)(\Omega_{ij})_{N_5}{}^{N_1}F^{N_2\ldots N_5}\\
S^{(5)}_{ij,k} &:= \frac{1}{2.4!}(\Gamma_{N_1\ldots N_6}\epsilon_k)(\Omega_{ij})^{N_1N_2}F^{N_3\ldots N_6}\\
S^{(6)}_{ij,k} &:= \frac{1}{4!}(\Gamma_{N_1}\epsilon_k)(\Sigma_{ij})_{N_2\ldots N_5}{}^{N_1}F^{N_2\ldots N_5}\\
S^{(7)}_{ij,k} &:= \frac{1}{2.3!}(\Gamma_{N_1N_2N_3}\epsilon_k)(\Sigma_{ij})_{N_4N_5N_6}{}^{N_1N_2}F^{N_3\ldots N_6}\\
S^{(8)}_{ij,k} &:= \frac{1}{4.3!}(\Gamma_{N_1\ldots N_5}\epsilon_k)(\Sigma_{ij})_{N_6N_7}{}^{N_1N_2N_3}F^{N_4\ldots N_7}\\
S^{(9)}_{ij,k} &:= \frac{1}{4!3!}(\Gamma_{N_1\ldots N_7}\epsilon_k)(\Sigma_{ij})_{N_8}{}^{N_1\ldots N_4}F^{N_5\ldots N_8}\\
S^{(10)}_{ij,k} &:= \frac{1}{5!4!}(\Gamma_{N_1\ldots N_9}\epsilon_k)(\Sigma_{ij})^{N_1\ldots N_5}F^{N_6\ldots N_9} \,.
\end{aligned} \tag{5.14}$$

Since the structures above form a basis, the Lie derivative can be expressed in terms of them. Indeed from (5.6) and (5.14) and using the Killing property of ϵ_k, one finds,

$$
\mathcal{L}_{K_{ij}}\epsilon_k = \frac{1}{6}S^{(1)}_{ij,k} + \frac{1}{12}S^{(2)}_{ij,k} - \frac{1}{3}S^{(3)}_{ij,k} + 10S^{(10)}_{ij,k}\,. \tag{5.15}
$$

By Fierz-rearranging (5.14) we get the following relations

$$
\begin{aligned}
S^{(1)}_{k(i,j)} &= -\frac{1}{16}S^{(1)}_{ij,k} - \frac{1}{8}S^{(2)}_{ij,k} + \frac{1}{2}S^{(4)}_{ij,k} - \frac{1}{4}S^{(5)}_{ij,k} + 15S^{(6)}_{ij,k} - \\
&\quad - \frac{15}{2}S^{(7)}_{ij,k} - \frac{15}{2}S^{(9)}_{ij,k} - 15S^{(10)}_{ij,k} \\
S^{(2)}_{k(i,j)} &= -\frac{7}{32}S^{(1)}_{ij,k} - \frac{5}{32}S^{(2)}_{ij,k} + \frac{7}{16}S^{(3)}_{ij,k} + \frac{5}{4}S^{(4)}_{ij,k} - \frac{3}{16}S^{(5)}_{ij,k} - \frac{75}{4}S^{(6)}_{ij,k} + \\
&\quad + \frac{45}{4}S^{(7)}_{ij,k} + \frac{15}{4}S^{(8)}_{ij,k} + \frac{15}{4}S^{(9)}_{ij,k} + \frac{45}{4}S^{(10)}_{ij,k} \\
2S^{(3)}_{k(i,j)} &= \frac{3}{16}S^{(2)}_{ij,k} - \frac{1}{8}S^{(3)}_{ij,k} - \frac{3}{8}S^{(5)}_{ij,k} + \frac{45}{2}S^{(6)}_{ij,k} + \frac{15}{2}S^{(8)}_{ij,k} + \frac{45}{2}S^{(10)}_{ij,k} \\
2S^{(4)}_{k(i,j)} &= \frac{7}{64}S^{(1)}_{ij,k} + \frac{5}{32}S^{(2)}_{ij,k} + \frac{5}{8}S^{(4)}_{ij,k} - \frac{3}{16}S^{(5)}_{ij,k} - \frac{75}{4}S^{(6)}_{ij,k} + \frac{45}{8}S^{(7)}_{ij,k} - \\
&\quad - \frac{15}{8}S^{(9)}_{ij,k} - \frac{45}{4}S^{(10)}_{ij,k} \\
2S^{(5)}_{k(i,j)} &= -\frac{21}{32}S^{(1)}_{ij,k} - \frac{9}{32}S^{(2)}_{ij,k} - \frac{21}{16}S^{(3)}_{ij,k} - \frac{9}{4}S^{(4)}_{ij,k} + \frac{1}{16}S^{(5)}_{ij,k} - \frac{135}{4}S^{(6)}_{ij,k} + \\
&\quad + \frac{15}{4}S^{(7)}_{ij,k} - \frac{45}{4}S^{(8)}_{ij,k} + \frac{45}{4}S^{(9)}_{ij,k} - \frac{15}{4}S^{(10)}_{ij,k} \\
5!S^{(6)}_{k(i,j)} &= \frac{7}{32}S^{(1)}_{ij,k} - \frac{5}{32}S^{(2)}_{ij,k} + \frac{7}{16}S^{(3)}_{ij,k} - \frac{5}{4}S^{(4)}_{ij,k} - \frac{3}{16}S^{(5)}_{ij,k} - \frac{75}{4}S^{(6)}_{ij,k} - \\
&\quad - \frac{45}{4}S^{(7)}_{ij,k} + \frac{15}{4}S^{(8)}_{ij,k} - \frac{15}{4}S^{(9)}_{ij,k} + \frac{45}{4}S^{(10)}_{ij,k} \\
5!S^{(7)}_{k(i,j)} &= -\frac{21}{16}S^{(1)}_{ij,k} + \frac{9}{8}S^{(2)}_{ij,k} + \frac{9}{2}S^{(4)}_{ij,k} + \frac{1}{4}S^{(5)}_{ij,k} - 135S^{(6)}_{ij,k} - \frac{15}{2}S^{(7)}_{ij,k} + \\
&\quad + \frac{45}{2}S^{(9)}_{ij,k} + 15S^{(10)}_{ij,k} \\
5!S^{(8)}_{k(i,j)} &= \frac{15}{16}S^{(2)}_{ij,k} + \frac{35}{8}S^{(3)}_{ij,k} - \frac{15}{8}S^{(5)}_{ij,k} + \frac{225}{2}S^{(6)}_{ij,k} - \frac{45}{2}S^{(8)}_{ij,k} + \frac{225}{2}S^{(10)}_{ij,k} \\
5!S^{(9)}_{k(i,j)} &= -\frac{35}{16}S^{(1)}_{ij,k} + \frac{5}{8}S^{(2)}_{ij,k} - \frac{5}{2}S^{(4)}_{ij,k} + \frac{5}{4}S^{(5)}_{ij,k} - 75S^{(6)}_{ij,k} + \frac{75}{2}S^{(7)}_{ij,k} - \\
&\quad - \frac{45}{2}S^{(9)}_{ij,k} + 75S^{(10)}_{ij,k} \\
5!S^{(10)}_{k(i,j)} &= -\frac{21}{32}S^{(1)}_{ij,k} + \frac{9}{32}S^{(2)}_{ij,k} + \frac{21}{16}S^{(3)}_{ij,k} - \frac{9}{4}S^{(4)}_{ij,k} - \frac{1}{16}S^{(5)}_{ij,k} + \frac{135}{4}S^{(6)}_{ij,k} + \\
&\quad + \frac{15}{4}S^{(7)}_{ij,k} + \frac{45}{4}S^{(8)}_{ij,k} + \frac{45}{4}S^{(9)}_{ij,k} + \frac{15}{4}S^{(10)}_{ij,k}\,.
\end{aligned} \tag{5.16}
$$

Symmetrizing all i, j, k indices, (5.16) reduces to a system of ten equations on ten unknowns $S^{(1)}_{(ij,k)}, \ldots S^{(10)}_{(ij,k)}$. It turns out however that only seven of the equations are linearly independent. Consequently, all structures can be expressed in terms of three independent ones which we may take to be $S^{(1)}_{(ij,k)}, \ldots S^{(3)}_{(ij,k)}$. This is in accordance to the fact that there are exactly three spinors in the decomposition of the tensor product of a four-form and the symmetrized product of thee spinors,

$$(00010) \otimes (00001)^{3\otimes_s} = 3(00001) \oplus \cdots . \tag{5.17}$$

Explicitly we have,

$$\begin{aligned}
S^{(4)}_{(ij,k)} &= \frac{1}{8}\left(3S^{(1)}_{(ij,k)} + 4S^{(2)}_{(ij,k)} - 4S^{(3)}_{(ij,k)}\right) \\
S^{(5)}_{(ij,k)} &= \frac{1}{2}\left(-2S^{(1)}_{(ij,k)} - S^{(2)}_{(ij,k)} - 2S^{(3)}_{(ij,k)}\right) \\
S^{(6)}_{(ij,k)} &= \frac{1}{120}\left(-S^{(2)}_{(ij,k)} + 2S^{(3)}_{(ij,k)}\right) \\
S^{(7)}_{(ij,k)} &= \frac{1}{120}\left(-S^{(1)}_{(ij,k)} + 4S^{(2)}_{(ij,k)} - 4S^{(3)}_{(ij,k)}\right) \\
S^{(8)}_{(ij,k)} &= \frac{1}{12}S^{(3)}_{(ij,k)} \\
S^{(9)}_{(ij,k)} &= -\frac{1}{24}S^{(1)}_{(ij,k)} \\
S^{(10)}_{(ij,k)} &= \frac{1}{120}\left(-2S^{(1)}_{(ij,k)} - S^{(2)}_{(ij,k)} + 4S^{(3)}_{(ij,k)}\right) .
\end{aligned} \tag{5.18}$$

From (5.18) and (5.15) we conclude that

$$\mathcal{L}_{K_{(ij}}\epsilon_{k)} = 0\,,$$

and therefore (5.12) is proven.

As a consequence, in the case where there is one Killing spinor one cannot generate another Killing spinor by taking the Lie derivative of the first one. In other words, one supersymmetry does not necessarily imply a second one.

A direct calculation using (5.6) and the Killing property of the spinors, reveals that

$$\begin{aligned}
\mathcal{L}_{K_{ij}}(\bar\epsilon_m \Gamma_M \epsilon_n) = &\frac{1}{3}F_{MN_1N_2N_3}\ \bar\epsilon_m \Gamma^{N_1N_2}\epsilon_n\ (K_{ij})^{N_3} + \\
&+\frac{1}{6}({}^*F)_{MN_1\ldots N_6}\ \bar\epsilon_m \Gamma^{N_1\ldots N_5}\epsilon_n (K_{ij})^{N_6} - (m,n) \leftrightarrow (i,j)\,,
\end{aligned} \tag{5.19}$$

which implies that the right-hand side of (5.9) is antisymmetric under $(m,n) \leftrightarrow (i,j)$ as of course it should. Consequently, the structure constants obey

$$f_{ij,m}{}^{(k}\delta^{l)}_n + f_{ij,n}{}^{(k}\delta^{l)}_m + (m,n) \leftrightarrow (i,j) = 0\,. \tag{5.20}$$

Contracting (5.20) with $\delta^m_k \delta^n_l$ and using (5.12) we obtain

$$f_{ij,q}{}^q = 0\,. \tag{5.21}$$

Finally, contracting (5.20) with δ_l^n and using (5.21) and (5.12) we obtain (5.4) with

$$h_{ij} = -\frac{1}{N+1} f_{iq,j}{}^{q} \,. \tag{5.22}$$

Note that h_{ij} is antisymmetric by virtue of (5.12) and (5.21).

As a consequence of (5.4), the Jacobi identity for the even generators T_{ij}, is automatically satisfied. Indeed, (5.3) and (5.4) imply that the commutators of the subalgebra $\mathcal{G}_0^s$ are

$$[T_{ij}, T_{mn}] = 4h_{(m|(i}T_{j)|n)} \,, \tag{5.23}$$

from which it follows that

$$[[T_{ij}, T_{mn}], T_{kl}] + \text{cyclic} = 0 \,. \tag{5.24}$$

As an example, let us analyze the algebra of isometries associated with Killing spinors, in the case where there are exactly two[8] Killing spinors ϵ_1, ϵ_2.

In view of (5.23) we have the following algebra

$$[K_{11}, K_{22}] = -4\alpha K_{12} \,; \qquad [K_{11}, K_{12}] = -2\alpha K_{11} \,; \qquad [K_{12}, K_{22}] = -2\alpha K_{22} \,, \tag{5.25}$$

where we have set $\alpha := h_{12} = -h_{21}$.

- If $\alpha = 0$, the algebra (5.25) is isomorphic to $u(1), \oplus^2 u(1)$ or $\oplus^3 u(1)$, according to the number of linearly independent Killing vectors.

- If $\alpha \neq 0$, we set

$$X_1 := -\frac{1}{2\alpha} K_{12} \,, \qquad X_2 := \frac{1}{4\alpha}(K_{11} + K_{22}) \,, \qquad X_3 := \frac{1}{4\alpha}(-K_{11} + K_{22}) \,, \tag{5.26}$$

  so that (5.25) takes the form

$$[X_1, X_2] = X_3 \,; \qquad [X_2, X_3] = X_1 \,; \qquad [X_3, X_1] = -X_2 \,. \tag{5.27}$$

  This algebra is isomorphic to $so(1,2;\mathbb{R})$.

More generally:

- In the case where the number of Killing spinors is even ($N = 2r$) and h_{ij} is non-degenerate, the algebra $\mathcal{G}_0^s$ is isomorphic to a real form of $sp(2r,\mathbb{C})$. Perhaps the easiest way to see this is to note that an explicit matrix realization of (5.23) is given by

$$(T_{ij})^l{}_k = \delta_j^l h_{ki} + \delta_i^l h_{kj} \,. \tag{5.28}$$

  It is then straightforward to verify that

$$(T_{ij}^{\mathrm{t}})_k{}^m h_{ml} + h_{km}(T_{ij})^m{}_l = 0 \,, \tag{5.29}$$

  which, for an invertible antisymmetric matrix h, is equivalent to showing that T_{ij} span a real form C_r of $sp(2r,\mathbb{C})$.[9] If h is degenerate one obtains a contraction of $\mathcal{G}_0^s$.

[8]In the case of a single Killing spinor there can be only one associated Killing vector and the algebra of isometries is isomorphic to $u(1)$.

[9]The example examined above is the special case $r = 1$. Indeed, the matrices $X_1 := \sigma_1$, $X_2 := -i\sigma_2$, $X_3 := \sigma_3$ generate the algebra (5.27) which, over the complex numbers, is isomorphic to $C_1 := A_1$.

- In the case where the number of Killing spinors is odd ($N = 2r+1$), h_{ij} is necessarily degenerate and without loss of generality we can take it to be of the form

$$h = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & h_{i'j'} & \\ 0 & & & \end{pmatrix}, \tag{5.30}$$

where $i', j' = 2, \dots 2r+1$. The algebra (5.23) becomes

$$\begin{aligned} [T_{1j}, T_{mn}] &= h_{mj} T_{1n} + h_{nj} T_{1m} \\ [T_{i'j'}, T_{m'n'}] &= 4h_{(m'|(i'} T_{j')|n')} \,. \end{aligned} \tag{5.31}$$

The generators $\{T_{1m} : m = 1, \dots, N\}$ span a Heisenberg algebra H_r with central generator T_{11}. H_r is an ideal of $\mathcal{G}_0^s$, while –as in the case examined above– the generators $\{T_{i'j'}\}$ form a subalgebra of $\mathcal{G}_0^s$ isomorphic to a real form C_r of $sp(2r, \mathbb{C})$ or a contraction thereof. The total algebra is the semidirect sum $\mathcal{G}_0^s = C_r \oplus_s H_r$.

6. Supersymmetric M-brane configurations

Planar brane probes in Minkowski space are associated with supersymmetry projectors. This is most easily seen using the kappa-symmetry projector of world-volume actions. It has been observed in [21] that kappa-symmetry projectors for all branes are associated with *hermitean* matrices Γ

$$\mathrm{tr}\,\Gamma = 0 \qquad \Gamma^2 = 1 \,. \tag{6.1}$$

In the case of planar M-branes, the Γ's are the elements in a Clifford algebra associated with the world-volume forms of the branes. The space of supersymmetry projectors has a ring structure inherited from that of the Clifford algebra multiplication and addition. A first test on whether a configuration of branes is supersymmetric is to consider it as a probe in Minkowski spacetime and find whether the projectors associated with the branes are compatible, i.e. whether they mutually commute and whether they have common eigenvalues. If they do, then the number of eigenspinors with eigenvalue one is the number of supersymmetries preserved by the configuration. Many such supersymmetric brane configurations have been found, see e.g. [22]–[26], in applications in string- and M-theory,

In an M-brane configuration let us define as 'linearly independent' branes, those branes whose world-volumes are associated with linearly independent elements in the Clifford algebra. To our knowledge, it is not known what is the maximal number of linearly independent branes that are allowed in a supersymmetric configuration. We shall show that in M-theory the maximal number is thirty one and that such a configuration preserves one supersymmetry. The Clifford algebra elements associated with the world-volume forms of an M-wave, $MW: \ 0\ 1$, an M2-brane, $M2: \ 0\ 1\ 2$, an M5-brane, $M5: \ 0\ 1\ 2\ 3\ 4\ 5$, and an M-theory KK-monopole, $MK: \ 0\ 1\ 2\ 3\ 4\ 5\ 6$, which we call them collectively M-branes, are

$$\Gamma^{01}, \qquad \Gamma^{012}, \qquad \Gamma^{012345}, \qquad \Gamma^{0123456}, \tag{6.2}$$

respectively. The numbers denote the world-volume directions of the M-branes. Denoting (6.2) collectively with Γ, the Killing spinors satisfy $\Gamma\epsilon = \epsilon$ and the supersymmetry projector is $\frac{1}{\sqrt{2}}(1_{32} + \Gamma)$. Observe that all the matrices in (6.2) are hermitean and satisfy (6.1). Since in addition are real, they are also symmetric. These properties imply that all the Clifford algebra elements (6.2) are in $sl(32, \mathbb{R})$. In addition, these properties imply that each element in (6.2) is non-degenerate and has eigenvalues ± 1 with equal number of positive and negative eigenvalues. Therefore to find the maximal number of linearly independent branes which can appear in a supersymmetric configuration, we have to determine the maximal number of mutually commuting elements in $sl(32, \mathbb{R})$ which are hermitean and satisfy (6.1). Since it is required that the projectors of the branes of a supersymmetric configuration commute, the associated world-volume forms of the branes lie in a Cartan subalgebra (CSA) of $sl(32, \mathbb{R})$. So the maximal number of linearly independent M-branes in a supersymmetric configuration is 31.

To show that the maximal number of linearly independent M-branes is precisely 31, we have to demonstrate that there is a basis in CSA of $sl(32, \mathbb{R})$ spanned by hermitean, traceless and non-degenerate matrices which square to identity. It can be arranged that all such matrices are diagonal

$$K(\lambda_1, \dots, \lambda_{32}) = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & & \vdots & & \\ 0 & 0 & 0 & \dots & \lambda_{32} \end{pmatrix}, \tag{6.3}$$

$\lambda_\alpha = \pm 1$, $\alpha = 1, \dots, 32$, and $\sum_{\alpha=1}^{32} \lambda_\alpha = 0$. Any such matrix K can be written as an element of the CSA of $gl(32, \mathbb{R})$. Indeed $K = y^a m_{a,a+1}$, where $m_{a,a+1}$ is the standard basis of the CSA of $gl(32, \mathbb{R})$ as in the appendix A provided that

$$\lambda_1 = y^1\,, \qquad \lambda_r = -y^{r-1} + y^r\,, \quad r = 2, \dots, 31, \qquad \lambda_{32} = -y^{31}\,. \tag{6.4}$$

Clearly, this equation can be solved as

$$y^r = \sum_{s=1}^{r} \lambda_s\,, r = 1, \dots, 31\,. \tag{6.5}$$

So K is in the CSA. Conversely, any basis vector $m_{a,a+1}$ of the CSA can be written in terms of matrices K. For example consider the basis vector $m_{1,2}$. This can be written as

$$\frac{1}{2}(K(1, -1, \lambda_3, \dots, \lambda_{32}) + K(1, -1, -\lambda_3, \dots, -\lambda_{32})) \tag{6.6}$$

and similarly for the rest. This shows that the maximal number of linearly independent M-branes in a supersymmetric configuration is 31.

It remains to show that such configuration preserves one supersymmetry. This can be easily seen by observing that λ_1 in all the matrices of a basis can be chosen to be $+1$. This is because if in one of the matrices K has $\lambda_1 = -1$, we can take $-K$ as a basis vector.

Thus the spinor

$$\epsilon = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{6.7}$$

is an eigenspinor of all the projectors with eigenvalue one. So the configuration preserves at least one supersymmetry. In addition there is no other eigenspinor which is linearly independent from ϵ and has the same eigenvalue. To see this, suppose that there is such an eigenspinor η. Without loss of generality, we can assume that

$$\eta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} . \tag{6.8}$$

In that case the K matrices which have both ϵ and η as eigenspinors, are of the form

$$K(1, 1, \lambda_3, \ldots, \lambda_{32}) . \tag{6.9}$$

However such matrices do not span a basis of the CSA of $sl(32, \mathbb{R})$. For example, it is not possible to construct the basis vector $m_{1,2}$. Therefore, the configuration with 31 linearly independent branes preserves one supersymmetry.

Given a configuration of 31 M-branes which preserves one-supersymmetry, there are another 30 configurations of 31 branes and anti-branes which also preserve one supersymmetry. These configurations are made from the branes that are involved in the original configuration after some of them are replaced by their anti-branes. This is because, as we have seen, the world-volume elements of the branes of the original configuration can be simultaneously diagonalized with eigenvalues ± 1. So any spinor ϵ with only one non-vanishing entry is an eigenspinor of all world-volume elements with eigenvalues ± 1. The eigenvalue $+1$ is associated with the branes and the eigenvalue -1 is associated with the anti-branes.

An example of a supersymmetric M-brane configuration with 31 linearly independent branes is

$$\begin{array}{lccccccccccc}
M2: & 0 & 1 & 2 & & & & & & & & \\
M2: & 0 & & & 3 & 4 & & & & & & \\
M2: & 0 & & & & & 5 & 6 & & & & \\
M2: & 0 & & & & & & & 7 & 8 & & \\
M2: & 0 & & & & & & & & & 9 & 10 \\
M5: & 0 & & 2 & & 4 & & 6 & & 8 & & 10 \\
M5: & 0 & & 2 & 3 & & 5 & & & 8 & & 10
\end{array}$$

$$
\begin{array}{r@{\;:\;}ccccccccccc}
M5 & 0 & & 2 & & 4 & 5 & & 7 & & & 10 \\
M5 & 0 & 1 & & & 4 & & 6 & 7 & & & 10 \\
M5 & 0 & 1 & & & 4 & 5 & & & 8 & & 10 \\
M5 & 0 & & 2 & 3 & & & 6 & 7 & & & 10 \\
M5 & 0 & & 2 & 3 & & 5 & & 7 & & 9 & \\
M5 & 0 & 1 & & & 4 & & 6 & & 8 & 9 & \\
M5 & 0 & 1 & & 3 & & & 6 & & 8 & & 10 \\
M5 & 0 & & 2 & & 4 & 5 & & & 8 & 9 & \\
M5 & 0 & & 2 & 3 & & & 6 & & 8 & 9 & \\
M5 & 0 & & 2 & & 4 & & 6 & 7 & & 9 & \\
M5 & 0 & 1 & & & 4 & 5 & & 7 & & 9 & \\
M5 & 0 & 1 & & 3 & & & 6 & 7 & & 9 & \\
M5 & 0 & 1 & & 3 & & 5 & & & 8 & 9 & \\
M5 & 0 & 1 & & 3 & & 5 & & 7 & & & 10 \\
MK & 0 & & & & & 5 & 6 & 7 & 8 & 9 & 10 \\
MK & 0 & & & 3 & 4 & & & 7 & 8 & 9 & 10 \\
MK & 0 & & & 3 & 4 & 5 & 6 & & & 9 & 10 \\
MK & 0 & & & 3 & 4 & 5 & 6 & 7 & 8 & & \\
MK & 0 & 1 & 2 & & & & & 7 & 8 & 9 & 10 \\
MK & 0 & 1 & 2 & & & 5 & 6 & & & 9 & 10 \\
MK & 0 & 1 & 2 & & & 5 & 6 & 7 & 8 & & \\
MK & 0 & 1 & 2 & 3 & 4 & & & & & 9 & 10 \\
MK & 0 & 1 & 2 & 3 & 4 & & & 7 & 8 & & \\
MK & 0 & 1 & 2 & 3 & 4 & 5 & 6 & & & &
\end{array}
\tag{6.10}
$$

One can verify that all the associated world-volume elements are mutually commuting, hermitean and satisfy (6.1). The above brane configuration preserves one supersymmetry. Observe that this example of supersymmetric M-brane configuration does not involve the M-theory pp-wave.

There is up to conjugation a unique CSA of $sl(32,\mathbb{R})$ which is spanned by hermitean traceless matrices. Therefore *all* the supersymmetric configurations of M-branes with 31 linearly independent branes can be constructed by taking *different bases* in the CSA of

$sl(32,\mathbb{R})$ which can be written in terms of hermitean matrices that satisfy (6.1). In general though such bases will not be associated with simple forms as in (6.10). It is expected that there are supersymmetric configurations with 31 M-branes that involve the M-theory pp-wave and the bound state of an M2-brane within an M5-brane. In the latter case, the supersymmetry projector is a linear combination of an M2- and an M5-brane projector [27]. This classifies all supersymmetric brane configurations with 31 linearly independent M-branes. It also classifies all supersymmetric brane configurations with less than 31 linearly independent M-branes. This is because all such configurations can be constructed from those with 31 branes by removing some of the linearly independent elements of the Clifford algebra.

The action of $\mathrm{SL}(32,\mathbb{R})$ does *not* preserve the space of supersymmetric M-brane configurations. To see this consider a supersymmetric configuration of 31 linearly independent M-branes as in (6.10), and let $\{\Gamma_I : I = 1, \dots, 31\}$ be the world-volume elements of these branes in the Clifford algebra. Next take

$$\Gamma'_I = A\Gamma_I A^{-1}\,, \tag{6.11}$$

where $A \in \mathrm{SL}(32,\mathbb{R})$. The matrices $\{\Gamma'_I : I + 1, \dots, 31\}$ satisfy (6.1). Requiring in addition that $\{\Gamma'_I : I + 1, \dots, 31\}$ be hermitean, we find that

$$\Gamma_I(A^{\mathrm{t}}A) = (A^{\mathrm{t}}A)\Gamma_I\,, \qquad I = 1, \dots, 31\,. \tag{6.12}$$

Since the matrix $A^{\mathrm{t}}A$ commutes with all the elements of the CSA of $sl(32,\mathbb{R})$, A^tA is in the maximal torus of $sl(32,\mathbb{R})$. Therefore we conclude that not all elements of $\mathrm{SL}(32,\mathbb{R})$ preserve the space of supersymmetric planar M-brane configurations. If $A \in \mathrm{SO}(32,\mathbb{R}) \subset \mathrm{SL}(32,\mathbb{R})$, then (6.12) is satisfied. So $\mathrm{SO}(32,\mathbb{R})$ preserves the space of supersymmetric planar M-brane configurations but $\mathrm{SL}(32,\mathbb{R})$ does not. If there are solutions of eleven-dimensional qsupergravity which preserve one supersymmetry some of them may have the interpretation of configurations above with 31 branes, see also [28].

Acknowledgments

We thank C. Hull for explaining to us the results of [4]. D.T. thanks the European Union, grant number HPRN-2000-00122, which includes the Department of Physics, Queen Mary College University of London, as a sub-contractor.

A. Some properties of $sl(32,\mathbb{R})$

The Lie algebra $sl(32,\mathbb{R})$ is isomorphic to $M^0_{32}(\mathbb{R})$. A basis of $sl(32,\mathbb{R})$ can be constructed as follows: Define $m_{a,b}$ to be the matrix which entry 1 at the a,b position and zero otherwise, i.e.

$$(m_{a,b})^c{}_d = \delta^c_a \delta_{bd}\,.$$

Then define a basis in $sl(32,\mathbb{R})$ as follows:

$$
\begin{aligned}
H_a &= m_{a,a} - m_{a+1,a+1}\,, \qquad a = 1,\ldots,31 \\
E^+_{a,b} &= m_{a,b}\,, \qquad a < b \\
E^-_{a,b} &= m_{a,b}\,, \qquad a > b\,.
\end{aligned} \tag{A.1}
$$

The H generators span a Cartan subalgebra of $sl(32,\mathbb{R})$, E^+ are the step generators along the positive roots while E^- are the step generators along the negative roots. The step generators along the simple roots are $E^+_a = E^+_{a,a+1}$ for $a = 1,\ldots,31$.

Because of (2.4), all the gamma matrices can be written in the m_{ab} basis. To make this explicit, consider the following basis of gamma matrices

$$
\begin{aligned}
\Gamma^0 &= e^0 \otimes \gamma_9\,, \qquad \Gamma^1 = e^1 \otimes \gamma_9\,, \\
\Gamma^2 &= e^2 \otimes \gamma_9\,, \qquad \Gamma^{i+2} = 1_2 \otimes \gamma^i\,, i = 1,\ldots,8\,,
\end{aligned} \tag{A.2}
$$

where $\{\gamma^i : i = 1,2,\ldots,8\}$ are the gamma matrices associated with Cliff$(\mathbb{R}^8)$, $\gamma_9 = \gamma^{12\ldots8}$ and

$$
e^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad e^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.3}
$$

It is known that Spin(8) admits two inequivalent real eight-dimensional chiral spinor representations. These combine to a sixteen-dimensional Majorana representation. An explicit expression for the γ^i matrices in the latter representation is

$$
\begin{aligned}
\gamma^1 &= e^1 \otimes e^0 \otimes I_1\,, \qquad \gamma^2 = e^1 \otimes e^0 \otimes I_2\,, \qquad \gamma^2 = e^1 \otimes e^0 \otimes I_3\,, \\
\gamma^4 &= e^1 \otimes e^0 \otimes J_1\,, \qquad \gamma^5 = e^1 \otimes e^0 \otimes J_2\,, \qquad \gamma^6 = e^1 \otimes e^0 \otimes J_3\,, \\
\gamma^7 &= 1_2 \otimes e^1 \otimes 1_4\,, \qquad \gamma^8 = 1_2 \otimes e^2 \otimes 1_4\,,
\end{aligned} \tag{A.4}
$$

where I_1, I_2, I_3 and J_1, J_2, J_3 are bases in $\Lambda^{2+}(\mathbb{R}^4)$ and $\Lambda^{2-}(\mathbb{R}^4)$, respectively. Explicitly, we have

$$
\begin{aligned}
(I_r)_{0s} &= \delta_{rs}\,, \qquad (I_r)_{st} = \epsilon_{rst}\,, \qquad r,s,t = 1,\ldots,3\,, \\
(J_r)_{0s} &= \delta_{rs}\,, \qquad (J_r)_{st} = -\epsilon_{rst}\,.
\end{aligned} \tag{A.5}
$$

It is straightforward to see that all the γ^i matrices are real, traceless and hermitean. All the gamma matrices $\{\Gamma^A; A = 0,\ldots,10\}$ can be expressed in terms of the m_{ab} basis of $sl(32,\mathbb{R})$. In particular, we have

$$
\Gamma_A = X^{ab}_A m_{ab} \tag{A.6}
$$

for some numerical coefficients X^{ab}_A. Consequently, all the products of gamma matrices can be written in the m_{ab} basis as

$$
\Gamma_{A_1A_2\ldots A_n} = X^{ab}_{A_1A_2\ldots A_n} m_{ab}\,, \tag{A.7}
$$

where again X are numerical coefficients. Conversely, (2.4) implies that we can express m_{ab} in terms of skew-symmetric products of gamma matrices. The above discussion in particular implies that the supercovariant curvature $\mathcal{R}$ can be written in terms of the m_{ab} basis.

B. Spinorial Lie derivatives and Killing-Yano tensors

It is well-known that given a Killing vector field, one can define a spinorial Lie derivative which acts as a derivation on the space of $\Lambda^*(M) \otimes S$. Here we shall define a spinorial Lie derivative with respect to a vector p-form, Y, i.e. a section of $T(M) \otimes \Lambda^p(M) = \Lambda^p_1(M)$.

It is well known that given $Y \in \Lambda^p_1(M)$, we can define a derivation on the space of forms as

$$\mathcal{L}_Y = i_Y d\omega + (-1)^p d i_Y \omega \,, \tag{B.1}$$

where $\omega \in \Lambda^q(M)$, d is the exterior derivative and

$$i_Y \omega = \frac{1}{p!(q-1)!} Y^b{}_{a_1 \ldots a_p} \omega_{b a_{p+1} \ldots a_{p+q-1}} dx^{a_1} \wedge \cdots \wedge dx^{a_{p+q-1}} \,. \tag{B.2}$$

In analogy with the spinorial derivative along a Killing vector, we take

$$\mathcal{L}_Y \epsilon = Y^b \nabla_b \epsilon + Q_{ab} \Gamma^{ab} \epsilon \,, \tag{B.3}$$

where we have suppressed the form indices of Y and Q is a section of Λ^p_2 to be determined. Demanding that

$$\mathcal{L}_Y(\Gamma\epsilon) = (-1)^p \Gamma \mathcal{L}_Y \epsilon \tag{B.4}$$

we find

$$(\nabla_a Y_{b,c_1,\ldots c_p} - 4 Q_{c_1 \ldots c_p, ab}) dx^{c_1} \wedge \ldots \wedge dx^{c_p} \wedge dx^a = 0 \,. \tag{B.5}$$

If $Y \in \Lambda^0_1$, we can take $Q_{ab} = \frac{1}{4} \nabla_a Y_b$. Since Q_{ab} is skew-symmetric in a, b, consistency requires that

$$\nabla_a Y_b + \nabla_b Y_a = 0 \tag{B.6}$$

and so Y is a Killing vector. If $Y \in \Lambda^1_1$, the most general solution of (B.5) is

$$Q_{c,ab} = \frac{1}{8} (\nabla_c Y_{a,b} - \nabla_c Y_{b,a} + \nabla_a Y_{c,b} - \nabla_b Y_{c,a} + \nabla_a Y_{b,c} - \nabla_b Y_{a,c}) \,, \tag{B.7}$$

where $Y_{a,b} = g_{ac} Y^c{}_b$. If $Y_{ab} = -Y_{ba}$, then

$$Q_{c,ab} = \frac{1}{4} \nabla_c Y_{ab} \,. \tag{B.8}$$

More generally, in order to solve (B.5), we define

$$T_{b,c_1 \ldots c_{p+1}} := \nabla_{[c_1|} Y_{b,|c_2 \ldots c_p]} \tag{B.9}$$

and expand in irreducible parts,

$$T_{b,c_1 \ldots c_{p+1}} = T^{(p+1,1)}_{b,c_1 \ldots c_{p+1}} + T^{(p+2,0)}_{bc_1 \ldots c_{p+1}} + (p+1) g_{b[c_1} T^{(p,0)}_{c_2 \ldots c_{p+1}]} \,, \tag{B.10}$$

where

$$\begin{aligned} T^{(p,0)}_{c_1 \ldots c_p} &= \frac{1}{D-p} \, T^f{}_{,fc_1 \ldots c_p} \\ T^{(p+2,0)}_{c_1 \ldots c_{p+2}} &= T_{[c_1, c_2 \ldots c_{p+2}]} \\ T^{(p+1,1)}_{b,c_1 \ldots c_{p+1}} &= T_{b,c_1 \ldots c_{p+1}} - T^{(p+2,0)}_{bc_1 \ldots c_{p+1}} - (p+1) g_{b[c_1} T^{(p,0)}_{c_2 \ldots c_{p+1}]} \,. \end{aligned} \tag{B.11}$$

Note that an irreducible (p,k)-tensor $V^{(p,k)}$ satisfies

$$V^{(p,k)}_{[b_1\dots b_p,c_1]\dots c_k} = g^{b_1 c_1} V^{(p,k)}_{b_1\dots b_p,c_1\dots c_k} = 0\,, \qquad p\geq k\,. \tag{B.12}$$

Similarly we expand,

$$\begin{aligned} Q_{ab,c_1\dots c_p} = & \; Q^{(p,2)}_{ab,c_1\dots c_p} + Q^{(p+2,0)}_{abc_1\dots c_p} + p(p-1)\; g_{a[c_1|} g_{b|c_2} Q^{(p-2,0)}_{c_3\dots c_p]} + \\ & + p\; \left(g_{a[c_1|} Q^{(p-1,1)}_{b,|c_2\dots c_p]} - g_{b[c_1|} Q^{(p-1,1)}_{a,|c_2\dots c_p]} \right) + \\ & + p\; \left(g_{a[c_1|} Q^{(p,0)}_{b|c_2\dots c_p]} - g_{b[c_1|} Q^{(p,0)}_{a|c_2\dots c_p]} \right) + \\ & + \left(Q^{(p+1,1)}_{a,bc_1\dots c_p} - Q^{(p+1,1)}_{b,ac_1\dots c_p} \right). \end{aligned} \tag{B.13}$$

Equation (B.5) is equivalent to

$$\begin{aligned} Q^{(p+2,0)} &= -\frac{1}{4}\, T^{(p+2,0)} \\ Q^{(p+1,1)} &= -\frac{p+1}{4p}\, T^{(p+1,1)} \\ Q^{(p,0)} &= \frac{p+1}{4p}\, T^{(p,0)}\,, \end{aligned} \tag{B.14}$$

Plugging (B.14) into (B.13), we get the most general solution to (B.5),

$$\begin{aligned} Q_{ab,c_1\dots c_p} = & -\frac{1}{2} T_{[a,b]c_1\dots c_p} + \frac{1}{4} T_{[c_1|,ab|c_2\dots c_p]} + \\ & + Q^{(p,2)}_{ab,c_1\dots c_p} + p(p-1)\; g_{a[c_1|} g_{b|c_2} Q^{(p-2,0)}_{c_3\dots c_p} + \\ & + p\left(g_{a[c_1|} Q^{(p-1,1)}_{b,|c_2\dots c_p]} - g_{b[c_1|} Q^{(p-1,1)}_{a,|c_2\dots c_p]} \right), \end{aligned} \tag{B.15}$$

where $Q^{(p,2)}$, $Q^{(p-2,0)}$, $Q^{(p-1,1)}$ are arbitrary. Note that for $p = 1$ the above expression reduces to (B.7).

If we set

$$Q_{c_1\dots c_p,ab} = \frac{1}{4} \nabla_a Y_{b,c_1\dots c_p}\,, \tag{B.16}$$

consistency requires that

$$\nabla_{(a} Y_{b),c_1\dots c_p} = 0\,, \tag{B.17}$$

so that Y is a Killing-Yano tensor.

The anticommutator of odd generators of the asymptotic supersymmetry algebra of a spacetime with M2- and M5-branes is [29] has central terms which are brane charges. One may be tempted to introduce similar terms in the symmetry superalgebra of a supersymmetric background,

$$\{Q_i, Q_j\} = T_{ij} + C_{ij} + G_{ij}\,, \tag{B.18}$$

where C and G are the analogues of the M2-and M5-brane charges respectively. If C and G are central elements, then the commutator of C and G with Q would have to vanish. These commutators may be geometrically computed using the spinorial Lie derivatives along the two- and five-forms associated with the Killing spinors. However a preliminary computation has shown that such spinorial Lie derivatives do not vanish.

JHEP07(2003)018

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